A lower bound for the eigenvalues of the Sen–Witten operator on closed spacelike hypersurfaces

László B Szabados
Research Institute for Particle and Nuclear Physics
H-1525 Budapest 114, P. O. Box 49, Hungary
e-mail: lbszab@rmki.kfki.hu

February 2, 2008

Abstract

The eigenvalue problem for the Sen–Witten operator on closed spacelike hypersurfaces is investigated. The (square of its) eigenvalues are shown to be given exactly by the 3-surface integral appearing in the expression of the total energy-momentum of the matter+gravity systems in Witten's energy positivity proof. A sharp lower bound for the eigenvalues, given in terms of the constraint parts of the spacetime Einstein tensor, i.e. the energy and momentum densities of the matter fields, is given.

1 Introduction

A promising approach of constructing observables of the gravitational field in general relativity could be based on the spectral analysis of the Dirac operators on various submanifolds of the spacetime. For example, the eigenvalues of these operators are such gauge invariant objects, which are expected to reflect the geometrical properties of the submanifold in question, e.g. in the form of some lower bound for the eigenvalues in terms of other well known geometrical objects. The first who gave such a lower bound was Lichnerowicz [1]: he showed, in particular, that on a closed Riemannian spin manifold Σ with positive scalar curvature $\frac{1}{4}\inf\{R(p)|p\in\Sigma\}$ is a lower bound for the square of the eigenvalues. However, this bound is not sharp: on a metric 2-sphere with radius r the (positive) eigenvalues are $\frac{n}{r}$, $n\in\mathbb{N}$, while on metric spheres the bounds were expected to be saturated. In fact, in the last two decades such *sharp* lower bounds were found in terms of the scalar curvature [2, 3, 4, 5] or the volume [6, 5, 7]. In particular, in dimension m the sharp lower bound, given by Friedrich [2, 5], is $\frac{m}{4(m-1)}\inf\{R(p)|p\in\Sigma\}$. Similar results exist for hypersurface Dirac operators when the lower bounds are given in terms of the curvature scalar of the intrinsic geometry and the trace of the extrinsic curvature [8].

To have significance of these results in general relativity we should be able to link the bounds to well known concepts of physics, e.g. the objects defined in a natural way on a spacelike hypersurface Σ of a Lorentzian 4-manifold. For example, the curvature scalar R of the intrinsic geometry of Σ, by means of which many of the bounds above were formulated, or the square of the trace χ of the extrinsic curvature of Σ in the spacetime, appearing in the bound given in [8], are not really '4-covariant'. They are only terms in the Hamiltonian constraint part of the spacetime Einstein tensor. Moreover, the sign of χ^2 in the bound given in [8] is *negative*, which *decreases* the lower bound, and hence its usefulness is questionable.

The aim of the present paper is to derive a sharp lower bound for the eigenvalues of the Sen–Witten operator (i.e. the Dirac operator built from the hypersurface Sen connection) on closed spacelike hypersurfaces of the spacetime, which bound has a clear physical interpretation. We give a new kind of lower bound, given in terms of the constraint parts of the four dimensional Einstein tensor, rather than the intrinsic scalar curvature or the square of the trace of the extrinsic curvature. Its physical significance comes from the fact that, through Einstein's equations, this is just the energy and momentum density of the matter fields, for which we have a certain form of a positivity requirement (dominant energy condition). We find that on a closed spacelike hypersurface Σ the eigenvalues of the Sen–Witten operator is given by the expression of the total energy of the matter+gravity systems appearing in Witten's positive energy proof. This provides a sharp lower bound for the eigenvalues: it is an average of the flux of the energy current of the matter fields seen by the null observers. Through the example of a $t=$ const hypersurface of the $k=1$ Friedman–Robertson–Walker spacetime we show that this bound is sharp.

We use the abstract index formalism and the sign conventions of [9]. In particular, the signature of the spacetime metric is $(+,-,-,-)$, the curvature and Ricci tensors and the curvature scalar are defined by $R^a{}_{bcd}X^b := -(\nabla_c\nabla_d - \nabla_d\nabla_c)X^a$, $R_{bd} := R^a{}_{bad}$ and $R := R_{ab}g^{ab}$, respectively. Then Einstein's equations take the form $G_{ab} = -\kappa T_{ab}$, where $\kappa := 8\pi G$ with Newton's gravitational constant G.

2 Geometrical preliminaries

2.1 Metrics on bundles over Σ

Let Σ be a smooth orientable spacelike hypersurface, t^a its future pointing unit normal, and define $P^a_b := \delta^a_b - t^a t_b$. This is the orthogonal projection to Σ, by means of which the induced (negative definite) 3-metric is defined by $h_{ab} := P^c_a P^d_b g_{cd}$. We assume that the spacetime is space and time orientable, at least on an open neighbourhood of Σ, in which case t^a can be (and, in what follows, will be) chosen to be globally defined.

Let $\mathbb{V}^a(\Sigma)$ denote the pull back to Σ of the spacetime tangent bundle, which decomposes in a unique way to the g_{ab}-orthogonal direct sum of the tangent

bundle $T\Sigma$ and the normal bundle spanned by t^a. g_{ab} is a Lorentzian fibre metric, and we call the triple $(\mathbb{V}^a(\Sigma), g_{ab}, P^a_b)$ the Lorentzian vector bundle over Σ. It is the projection P^a_b as a base point preserving bundle endomorphism which tells us how the tangent bundle $T\Sigma$ is embedded in $\mathbb{V}^a(\Sigma)$. Since both $T\Sigma$ and the normal bundle of Σ in M are globally trivializable, $\mathbb{V}^a(\Sigma)$ is also. This implies the existence of a spinor structure too. Let $\mathbb{S}^A(\Sigma)$ denote the bundle of 2-component (i.e. Weyl) spinors over Σ, and we denote the complex conjugate bundle by $\bar{\mathbb{S}}^{A'}(\Sigma)$. As is usual in general relativity (see e.g. [9]), we identify the Hermitian subbundle of $\mathbb{S}^A(\Sigma) \otimes \bar{\mathbb{S}}^{A'}(\Sigma)$ with $\mathbb{V}^a(\Sigma)$. Thus we can convert tensor indices to pairs of spinor indices and vice versa freely.

On the spinor bundle two metrics are defined: The first is the natural symplectic metric ε_{AB}, while the other is the positive definite Hermitian metric $G_{AB'} := \sqrt{2} t_{AB'}$. (The reason of the factor $\sqrt{2}$ is that for this definition $G^{AB'}$, the inverse of $G_{AB'}$ defined by $G^{AB'} G_{BB'} = \delta^A_B$, is just the contravariant form $\varepsilon^{AC} \varepsilon^{B'D'} G_{CD'}$ of the Hermitian metric, i.e. the Hermitian and the symplectic metrics are compatible.) The Hermitian metric defines the $\mathbb{C}$-linear bundle isomorphisms $\bar{\mathbb{S}}^{A'}(\Sigma) \to \mathbb{S}^A(\Sigma) : \bar{\lambda}^{A'} \mapsto -G^A{}_{A'} \bar{\lambda}^{A'}$ and $\bar{\mathbb{S}}_{A'}(\Sigma) \to \mathbb{S}_A(\Sigma) : \bar{\lambda}_{A'} \mapsto G_A{}^{A'} \bar{\lambda}_{A'}$; as well as

$$\langle \lambda_A, \phi_A \rangle := \int_\Sigma G^{AA'} \lambda_A \bar{\phi}_{A'} \mathrm{d}\Sigma, \tag{2.1}$$

which is a global L_2 scalar product on the space of the (square integrable) spinor fields on Σ. This defines a norm in the standard way: $\|\lambda_A\|^2 := \langle \lambda_A, \lambda_A \rangle$.

2.2 The Sen connection

The intrinsic Levi-Civita covariant derivative operator, defined on $T\Sigma$, will be denoted by D_e. This will be extended to the whole $\mathbb{V}^a(\Sigma)$ by requiring $D_e t_a = 0$. We introduce another connection on $\mathbb{V}^a(\Sigma)$, the so-called Sen connection [10] by $\mathcal{D}_a := P^b_a \nabla_b$. Clearly, both D_e and $\mathcal{D}_e$ annihilate the fiber metric g_{ab}, but the projection is annihilated only by D_e. (In the language of [11] D_e is a reduction of $\mathcal{D}_e$, and the reduction is made by requiring that the projection be annihilated by the covariant derivative operator.) The extrinsic curvature of Σ in M is $\chi_{ab} := \mathcal{D}_a t_b = \chi_{(ab)}$. In terms of D_e and the extrinsic curvature the action of the Sen derivative on an arbitrary cross section X^a of $\mathbb{V}^a(\Sigma)$ is given by

$$\mathcal{D}_e X^a = D_e X^a + \big(\chi_e{}^a t_b - t^a \chi_{eb}\big) X^b. \tag{2.2}$$

The curvature of $\mathcal{D}_a$ is defined by the convention $-F^a{}_{bcd} X^b v^c w^d := v^c \mathcal{D}_c(w^d \mathcal{D}_d X^a) - w^c \mathcal{D}_c(v^d \mathcal{D}_d X^a) - [v,w]^e \mathcal{D}_e X^a$ for any X^a and v^c and w^c tangent to Σ. This is just the pull back to Σ of the spacetime curvature 2-form, $F^a{}_{bcd} = {}^4R^a{}_{bef} P^e_c P^f_d$, and it can be re-expressed as

$$\begin{aligned} F_{abcd} &= R_{abcd} + \chi_{ac}\chi_{bd} - \chi_{ad}\chi_{bc} + \\ &= t_a\big(D_c \chi_{db} - D_d \chi_{cb}\big) - t_b\big(D_c \chi_{da} - D_d \chi_{ca}\big), \end{aligned} \tag{2.3}$$

where R_{abcd} is the curvature tensor of the intrinsic geometry of (Σ, h_{ab}).

$\mathcal{D}_e$ extends in a natural way to the spinor bundle, and its action on a spinor field is

$$\mathcal{D}_e\lambda_A = D_e\lambda_A - \chi_{eAA'}t^{A'}{}_B\lambda^B. \tag{2.4}$$

The commutator of two Sen operators acting on the spinor filed λ^A is

$$\big(\mathcal{D}_c\mathcal{D}_d - \mathcal{D}_d\mathcal{D}_c\big)\lambda^A = -F^A{}_{Bcd}\lambda^B - 2\chi^e{}_{[c}t_{d]}\mathcal{D}_e\lambda^A, \tag{2.5}$$

where the curvature $F^A{}_{Bcd}$ is just the pull back to Σ of the anti-self-dual part of the spacetime curvature 2-form, which can also be expressed by the (spinor form of the) intrinsic curvature and the extrinsic curvature. For an introduction of the Sen connection not using the embedding of Σ in M, see [12].

The Sen–Witten operator, i.e. the Dirac operator built from the Sen connection, is defined to be $\mathcal{D} : \mathbb{S}^A(\Sigma) \to \bar{\mathbb{S}}_{A'} : \lambda^A \mapsto \mathcal{D}_{A'A}\lambda^A$. Since

$$\langle \mathcal{D}_{A'A}\lambda^A, \bar\phi_{B'}\rangle = \int_\Sigma D_{AA'}\big(\lambda^A G^{A'B}\phi_B\big)\mathrm{d}\Sigma + \int_\Sigma \lambda^A G_{AA'}\big(\mathcal{D}^{A'B}\phi_B\big)\mathrm{d}\Sigma,$$

the formal adjoint of $\mathcal{D}$ is $\mathcal{D}^* : \bar{\mathbb{S}}_{A'}(\Sigma) \to \mathbb{S}^A(\Sigma) : \bar\phi_{A'} \mapsto \mathcal{D}^{AA'}\bar\phi_{A'}$, i.e. essentially the complex conjugate of the Sen–Witten operator itself. Therefore, both $\mathcal{D}^*\mathcal{D} : \lambda^A \mapsto \mathcal{D}^{AA'}\mathcal{D}_{A'B}\lambda^B$ and $\mathcal{D}\mathcal{D}^* : \bar\phi_{A'} \mapsto \mathcal{D}_{A'A}\mathcal{D}^{AB'}\bar\phi_{B'}$ are formally self-adjoint and they are essentially complex conjugate of each other. Moreover, since

$$\begin{aligned}\langle \mathcal{D}^{AA'}\mathcal{D}_{A'B}\lambda^B, \phi^C\rangle &= \int_\Sigma G_{AA'}\big(\mathcal{D}^A{}_{B'}\bar\phi^{B'}\big)\big(\mathcal{D}^{A'}{}_B\lambda^B\big)\mathrm{d}\Sigma + \\ &+ \int_\Sigma D_{AA'}\Big(\big(\mathcal{D}^{A'}{}_B\lambda^B\big)G^A{}_{B'}\bar\phi^{B'}\Big)\mathrm{d}\Sigma,\end{aligned} \tag{2.6}$$

for *closed* Σ the operator $\mathcal{D}^*\mathcal{D}$ is positive: $\langle \mathcal{D}^{AA'}\mathcal{D}_{A'B}\lambda^B, \lambda^C\rangle \geq 0$ for every spinor field λ^A.

2.3 The Sen–Witten identity

Using the commutator (2.5), the square of the Sen–Witten operator can be written as

$$\begin{aligned}\mathcal{D}_A{}^{A'}\mathcal{D}_{A'B}\lambda^B &= \mathcal{D}_{(A}{}^{A'}\mathcal{D}_{B)A'}\lambda^B + \frac{1}{2}\varepsilon_{AB}\mathcal{D}_R{}^{R'}\mathcal{D}_{R'}{}^B\lambda^B = \\ &= -\frac{1}{2}\varepsilon^{A'B'}\big(\mathcal{D}_{AA'}\mathcal{D}_{BB'} - \mathcal{D}_{BB'}\mathcal{D}_{AA'}\big)\lambda^B + \frac{1}{2}\mathcal{D}_e\mathcal{D}^e\lambda_A = \\ &= \frac{1}{2}\mathcal{D}_e\mathcal{D}^e\lambda_A + \frac{1}{2}\varepsilon^{A'B'}F^B{}_{CAA'BB'}\lambda^C + \varepsilon^{A'B'}\chi^e{}_{[a}t_{b]}\mathcal{D}_e\lambda_A.\end{aligned} \tag{2.7}$$

The last term can also be written as $\chi^e{}_{AA'}t^{A'}{}_B\mathcal{D}_e\lambda^B$. Using (2.3) and the fact that in three dimensions the curvature tensor can be expressed by the metric h_{ab} and the corresponding Ricci tensor and curvature scalar, a straightforward computation yields that

$$\varepsilon^{A'B'}F^B{}_{CAA'BB'} = -\frac{1}{4}\varepsilon_{AC}\bigl(R+\chi^2-\chi_{de}\chi^{de}\bigr)+\bigl(D_e\chi^e{}_{AA'}-D_{AA'}\chi\bigr)t^{A'}{}_A. \quad (2.8)$$

However, the terms on the right hand side are precisely the constraint parts of the spacetime Einstein tensor:

$$\frac{1}{2}\bigl(R+\chi^2-\chi_{ab}\chi^{ab}\bigr) = -{}^4G_{ab}t^at^b = \kappa T_{ab}t^at^b =: \kappa\mu, \quad (2.9)$$

$$\bigl(D_a\chi^a{}_b - D_b\chi\bigr) = -{}^4G_{ae}t^aP^e_b = \kappa T_{ae}t^aP^e_b := \kappa J_b; \quad (2.10)$$

where we used Einstein's field equations. The right hand side of these formulae define the energy density and the spatial momentum density of the matter fields, respectively, seen by the observer t^a. We will assume that the matter fields satisfy the dominant energy condition, i.e. $\mu^2 \geq |J_aJ^a|$. Substituting (2.8), (2.9) and (2.10) into (2.7) finally we obtain

$$\begin{aligned} 2\mathcal{D}^{AA'}\mathcal{D}_{A'B}\lambda^B &= \mathcal{D}_e\mathcal{D}^e\lambda^A + 2\chi^{eA}{}_{A'}t^{A'}{}_B\mathcal{D}_e\lambda^B - \\ &- \frac{1}{2}t_e{}^4G^{ef}t_f\lambda^A + \frac{1}{2}t_e{}^4G^{ef}P^{AA'}_f 2t_{A'B}\lambda^B. \end{aligned} \quad (2.11)$$

This equation is analogous to the Lichnerowicz identity [1]: The square of the Dirac operator is expressed in terms of the Laplacian and the curvature, but here $\mathcal{D}_e\mathcal{D}^e$ is *not* the intrinsic Laplacian and, in addition, the first derivative of the spinor field also appears on the right. Moreover, the curvature in (2.11) is not simply the scalar curvature, but a genuine tensorial piece of that. If, on the other hand, the extrinsic curvature is vanishing, then $\mathcal{D}_e$ reduces to the Levi-Civita D_e, and (2.11) reduces to $2D^{AA'}D_{A'B}\lambda^B = D_eD^e\lambda^A + \frac{1}{4}R$, which is the genuine Lichnerowicz identity for the three dimensional intrinsic Dirac operator. It might be interesting to note that the analogous identity for the Sen–Witten type operators on two (or more) codimensional submanifolds still does *not* reduce to the genuine Lichnerowicz identity even if the extrinsic curvatures are vanishing, because the reduced connection may still have non-trivial curvature in the normal bundle. For the example of spacelike 2-surfaces in Lorentzian spacetimes, see [13].

Contracting (2.11) with $t_{AB'}\bar\phi^{B'}$ and using the definitions, equation (2.4) and the fact that $G^A{}_{B'}G^{A'}{}_B$ acts as $-P^a_b$ on vectors tangent to Σ, we obtain

$$\begin{aligned} &D_{AA'}\bigl(2t^A{}_{B'}\bar\phi^{B'}\mathcal{D}^{A'}{}_B\lambda^B\bigr) + 2t^{AA'}\bigl(\mathcal{D}_{A'B}\lambda^B\bigr)\bigl(\mathcal{D}_{AB'}\bar\phi^{B'}\bigr) = \quad (2.12) \\ &= D_a\bigl(\bar\phi^{B'}t_{B'B}\mathcal{D}^a\lambda^B\bigr) - t_{AA'}\bigl(\mathcal{D}_e\lambda^A\bigr)\bigl(\mathcal{D}^e\bar\phi^{A'}\bigr) - \frac{1}{2}t^a{}^4G_{aBB'}\lambda^B\bar\phi^{B'}. \end{aligned}$$

Writing the total divergences in a different way we get the Reula–Tod (or the $SL(2,\mathbb{C})$ spinor) form [14] of the Sen–Witten identity:

$$D_a\big(t^{A'B}\bar\phi^{B'}\mathcal{D}_{BB'}\lambda^A-\bar\phi^{A'}t^{AB'}\mathcal{D}_{B'B}\lambda^B\big)+2t^{AA'}\big(\mathcal{D}_{A'B}\lambda^B\big)\big(\mathcal{D}_{AB'}\bar\phi^{B'}\big)= \\ =-t_{AA'}h^{ef}\big(\mathcal{D}_e\lambda^A\big)\big(\mathcal{D}_f\bar\phi^{A'}\big)-\frac{1}{2}t^a\,{}^4G_{aBB'}\lambda^B\bar\phi^{B'}. \tag{2.13}$$

Clearly, its right hand side is positive definite for $\lambda^A=\phi^A$ and matter fields satisfying the dominant energy condition. This identity is the basis of (probably the simplest) proof of the positivity of the ADM and Bondi–Sachs energies. (For the original proofs using Dirac spinors, see [15, 16], and for its extension to include black holes, see [17, 14].) The basic idea is that if Σ is asymptotically flat and $\lambda^A=\phi^A$ is chosen to be an asymptotically constant solution to the Sen-Witten equation $\mathcal{D}_{A'A}\lambda^A=0$, then the second term on the left is vanishing, and then, taking the integral of (2.13) and converting the total divergence to a 2-surface integral at infinity, the left hand side gives the ${}_0\lambda^A{}_0\bar\lambda^{A'}$-component of the ADM energy-momentum, where ${}_0\lambda^A$ is the asymptotic value of the spinor field λ^A. (At null infinity λ^A cannot be required to be asymptotically constant, only a weaker boundary condition may be imposed. For the details see [14].)

3 The eigenvalue problem for the Sen–Witten operators

According to the general theory of spinors (see e.g. the appendix of [21]) in three dimensions the spinors have two components, moreover the Sen–Witten operator maps cross sections of $\mathbb{S}^A(\Sigma)$ to cross sections of the complex conjugate bundle $\bar{\mathbb{S}}_{A'}(\Sigma)$, it seems natural to define the eigenvalue problem by

$$\mathrm{i}G_A{}^{A'}\mathcal{D}_{A'}{}^B\psi_B=-\frac{1}{\sqrt{2}}\beta\psi_A. \tag{3.1}$$

The unitary spinor form [18, 19] of (3.1), namely $\mathrm{i}\mathcal{D}_A{}^B\psi_B=-\frac{1}{\sqrt{2}}\beta\psi_A$, apparently makes this definition of the eigenvalue problem reasonable. (The choice for the apparently ad hoc coefficient $-1/\sqrt{2}$ in front of the eigenvalue β yields the compatibility with the known standard results in special cases.) However, it is desirable that the Hermitian metric be compatible with the connection in the sense that $\mathcal{D}_eG_{AA'}=0$. Unfortunately, since $\mathcal{D}_eG_{AA'}$ is $\sqrt{2}$-times the extrinsic curvature of Σ, in general this requirement cannot be satisfied. As a consequence, in general the eigenvalue β is *not* real. In fact, a straightforward calculation (by elementary integration by parts) gives that

$$\beta\|\psi_A\|^2=\bar\beta\|\psi_A\|^2+\mathrm{i}\int_\Sigma\chi G_{AA'}\psi^A\bar\psi^{A'}\mathrm{d}\Sigma+\mathrm{i}\sqrt{2}\int_\Sigma D_{AA'}\big(\psi^A\bar\psi^{A'}\big)\mathrm{d}\Sigma. \tag{3.2}$$

This implies that, even if Σ is closed, which will be assumed in the rest of this paper, the imaginary part of β is proportional to the integral of mean curvature χ weighted by the pointwise norm $G_{AA'}\psi^A\bar\psi^{A'}$, which is not zero in general.

This difficulty raises the question whether we can find a slightly different definition of the eigenvalue problem for the Sen–Witten operator yielding *real* eigenvalues. To motivate this, observe that although the base manifold Σ is only three dimensional, the connection $\mathcal{D}_e$ is four dimensional in its spirit, as originally it is defined on the Lorentzian vector bundle $\mathbb{V}^a(\Sigma)$. Since its fibres are four dimensional, the corresponding spinors are the four component Dirac spinors. Hence we should define the eigenvalue problem for the Sen–Witten operator in terms of the Dirac spinors.

Recall that a Dirac spinor Ψ^α is a pair of Weyl spinors λ^A and $\bar\mu^{A'}$, written them as a column vector

$$\Psi^\alpha = \begin{pmatrix} \lambda^A \\ \bar\mu^{A'} \end{pmatrix} \tag{3.3}$$

and adopting the convention $\alpha = A \oplus A'$, $\beta = B \oplus B'$ etc. Its derivative $\mathcal{D}_e\Psi^\alpha$ is the column vector consisting of $\mathcal{D}_e\lambda^A$ and $\mathcal{D}_e\bar\mu^{A'}$. If Dirac's γ-'matrices' are denoted by $\gamma^\alpha_{e\beta}$, then one can consider the eigenvalue problem

$$\mathrm{i}\gamma^\alpha_{e\beta}\mathcal{D}^e\Psi^\beta = \alpha\Psi^\alpha. \tag{3.4}$$

Explicitly, with the representation

$$\gamma^\alpha_{e\beta} = \sqrt{2}\begin{pmatrix} 0 & \varepsilon_{E'B'}\delta^A_E \\ \varepsilon_{EB}\delta^{A'}_{E'} & 0 \end{pmatrix} \tag{3.5}$$

(see e.g. [9], pp 221), this is just the pair of equations

$$\mathrm{i}\mathcal{D}_{A'}{}^A\lambda_A = -\frac{\alpha}{\sqrt{2}}\bar\mu_{A'}, \qquad \mathrm{i}\mathcal{D}_A{}^{A'}\bar\mu_{A'} = -\frac{\alpha}{\sqrt{2}}\lambda_A. \tag{3.6}$$

These imply that both the unprimed and the primed Weyl spinor parts of Ψ^α are eigenspinors of the square of the Sen–Witten operator with the *same* eigenvalue:

$$2\mathcal{D}^{AA'}\mathcal{D}_{A'B}\lambda^B = \alpha^2\lambda^A, \qquad 2\mathcal{D}^{A'A}\mathcal{D}_{AB'}\bar\mu^{B'} = \alpha^2\bar\mu^{A'}. \tag{3.7}$$

Then by (2.6) $0 \leq 2\langle\mathcal{D}^{AA'}\mathcal{D}_{A'B}\lambda^B, \lambda^C\rangle = \alpha^2\|\lambda^A\|$, i.e. *the eigenvalues α are real*. Conversely, if the pair (α^2, λ^A) is a solution of the eigenvalue problem for $2\mathcal{D}^*\mathcal{D}$ with nonzero α, then $(\pm\alpha, \Psi^\alpha_\pm)$ with $\bar\mu^{A'} := \mp(\sqrt{2}/\alpha)\mathrm{i}\mathcal{D}^{A'A}\lambda_A$ are solutions of the eigenvalue problem (3.4). Therefore, it is enough to study the eigenvalue problem for the second order operator $2\mathcal{D}^*\mathcal{D}$.

By (3.3) $\Psi^\alpha = (\lambda^A, \bar\mu^{A'})$ is a Dirac eigenspinor with eigenvalue α precisely when $(\lambda^A, -\bar\mu^{A'})$ is a Dirac eigenspinor with eigenvalue $-\alpha$. In the language of Dirac spinors this is formulated in terms of the chirality, represented by the so-called 'γ_5-matrix', denoted here by

$$\eta^\alpha{}_\beta := \frac{1}{4!}\varepsilon^{abcd}\gamma^\alpha_{a\mu}\gamma^\mu_{b\nu}\gamma^\nu_{c\rho}\gamma^\rho_{d\beta} = \mathrm{i}\begin{pmatrix} \delta^A_B & 0 \\ 0 & -\delta^{A'}_{B'} \end{pmatrix} \tag{3.8}$$

(see appendix II. of [21]). Since this is anti-commuting with $\gamma^{\alpha}_{e\beta}$, from (3.4) we obtain that $\mathrm{i}\gamma^{\alpha}_{e\mu}\mathcal{D}^e(\eta^{\mu}{}_{\beta}\Psi^{\beta}) = -\alpha(\eta^{\alpha}{}_{\beta}\Psi^{\beta})$. Thus if Ψ^{α} is a Dirac eigenspinor with eigenvalue α, then, in fact, $\eta^{\alpha}{}_{\beta}\Psi^{\beta}$ is a Dirac eigenspinor with eigenvalue $-\alpha$.

On the other hand, if there are Dirac eigenspinors with definite chirality, then they belong to the kernel of the Sen–Witten operator. Indeed, Dirac spinors with definite chirality have the structure either $(\lambda^A, 0)$ or $(0, \bar{\mu}^{A'})$, which, by (3.6), yield that $\mathcal{D}_{A'A}\lambda^A = 0$ or $\mathcal{D}_{AA'}\bar{\mu}^{A'} = 0$, respectively. Therefore, this notion of chirality cannot be used to decompose the space of the eigenspinors with given eigenvalue. Its role is simply to take a Dirac eigenspinor with eigenvalue α to a Dirac eigenspinor with eigenvalue $-\alpha$.

By the reality of the eigenvalues both the complex conjugate of the unprimed spinor part λ^A and the primed spinor part $\bar{\mu}^{A'}$ of Ψ^{α} are eigenspinors of $2\mathcal{D}\mathcal{D}^*$ with the same eigenvalue α^2. This raises the question as whether the eigenvalue problem can be restricted by $\lambda^A = \mu^A$, i.e. by requiring the Dirac eigenspinors Ψ^{α} to be Majorana spinors. However, (3.6) implies that in this case α would have to be purely imaginary or zero, i.e. the Sen–Witten operator does not have genuine, non-trivial Majorana eigenspinors.

Finally suppose that the extrinsic curvature is vanishing. In this special case $\mathcal{D}_e = D_e$, and let us consider the eigenvalue problem defined by (3.1). Then $\mathrm{i}G_A{}^{A'}D_{A'}{}^B(\mathrm{i}G_B{}^{B'}D_{B'}{}^C\psi_C) = \frac{1}{2}\beta^2\psi_A$. However, by $D_eG_{AA'} = 0$ we can write

$$\begin{aligned}\beta^2\psi_A &= -2G_A{}^{A'}G^B{}_{B'}D_{A'B}\big(D^{B'C}\psi_C\big) = -2G_A{}^{A'}G_{A'}{}^BD_{BB'}\big(D^{B'C}\psi_C\big) = \\ &= -2G_{AA'}G^{A'B}\big(D_B{}^{B'}D_{B'}{}^C\psi_C\big) = -2D_A{}^{A'}D_{A'}{}^B\psi_B.\end{aligned}$$

Thus the pair (β, ψ^A) is a solution of the eigenvalue problem for D^*D, and hence we may write $\beta = \alpha$ and $\psi^A = \lambda^A$. Then $\alpha\bar{\mu}_{A'} = -\mathrm{i}\sqrt{2}D_{A'}{}^A\lambda_A = \mathrm{i}\sqrt{2}G_{A'}{}^AG_A{}^{B'}D_{B'}{}^B\lambda_B = \mathrm{i}\sqrt{2}G_{A'}{}^A(\frac{1}{\sqrt{2}}\mathrm{i}\alpha\lambda_A) = \alpha G_{A'A}\lambda^A$; i.e. the primed spinor part $\bar{\mu}_{A'}$ of the Dirac eigenspinor is just $G_{A'A}\lambda^A$. Therefore, in the special case of the vanishing extrinsic curvature the eigenvalue problems (3.1) and (3.4) coincide.

4 Lower bounds for the eigenvalues

Suppose that λ^A is an eigenspinor of $2\mathcal{D}^*\mathcal{D}$ with eigenvalue α^2. Then since we assumed that Σ is closed, (2.13) yields that

$$\begin{aligned}\alpha^2\|\lambda^A\|^2 &= 2\sqrt{2}\int_{\Sigma}\big(t_{A'A}\bar{\lambda}^{A'}\mathcal{D}^{AB'}\mathcal{D}_{B'B}\lambda^B\big)\mathrm{d}\Sigma = \\ &= \sqrt{2}\int_{\Sigma}\Big(-t_{AA'}\big(\mathcal{D}_e\lambda^A\big)\big(\mathcal{D}^e\bar{\lambda}^{A'}\big) - \frac{1}{2}t^{a\ 4}G_{aBB'}\lambda^B\bar{\lambda}^{B'}\Big)\mathrm{d}\Sigma.\end{aligned} \tag{4.1}$$

This gives a lower bound for the eigenvalue α^2:

$$\alpha^2 \geq -\frac{1}{\sqrt{2}\|\lambda^A\|^2}\int_\Sigma t^a\,{}^4G_{aBB'}\lambda^B\bar\lambda^{B'}\,\mathrm{d}\Sigma \geq -\frac{1}{2}\inf\frac{\int_\Sigma t^a\,{}^4G_{ab}l^b\,\mathrm{d}\Sigma}{\int_\Sigma t_bl^b\,\mathrm{d}\Sigma},$$

where the infimum is taken on the set of the smooth, future pointing null vector fields l^a on Σ. However, this bound is certainly *not* sharp: In the special case of the vanishing extrinsic curvature the nominator is the integral of $-\frac{1}{2}Rt_al^a$ (see equations (2.9)-(2.10)), yielding Lichnerowicz's bound $\frac{1}{4}\inf\{R(p)|p\in\Sigma\}$ instead of Friedrich's sharp bound $\frac{3}{8}\inf\{R(p)|p\in\Sigma\}$.

To find the sharp bound, we follow the general philosophy of [2, 5] (see also [8]) and consider the modified Sen connection

$$\tilde{\mathcal{D}}_e\lambda^A := \mathcal{D}_e\lambda^A + sP_e^{AA'}\mathcal{D}_{A'B}\lambda^B \tag{4.2}$$

for some real constant s. Then a straightforward calculation gives

$$\begin{aligned}t_{AA'}\big(\tilde{\mathcal{D}}_e\lambda^A\big)\big(\tilde{\mathcal{D}}^e\bar\lambda^{A'}\big) + 2s(1+\frac{s}{4})t^{AA'}\big(\mathcal{D}_{AB'}\bar\lambda^{B'}\big)\big(\mathcal{D}_{A'B}\lambda^B\big) &=\\ = t_{AA'}\big(\mathcal{D}_e\lambda^A\big)\big(\mathcal{D}^e\bar\lambda^{A'}\big).&\end{aligned}$$

Using this expression for $t_{AA'}(\mathcal{D}_e\lambda^A)(\mathcal{D}^e\bar\lambda^{A'})$ in (4.1) we obtain

$$(1+s+\frac{3}{4}s^2)\alpha^2\|\lambda^A\|^2 = \sqrt{2}\int_\Sigma\Big(-t_{AA'}\big(\tilde{\mathcal{D}}_e\lambda^A\big)\big(\tilde{\mathcal{D}}^e\bar\lambda^{A'}\big) - \frac{1}{2}t^a\,{}^4G_{aBB'}\lambda^B\bar\lambda^{B'}\Big)\mathrm{d}\Sigma. \tag{4.3}$$

Its left hand has a minimum at $s=-\frac{2}{3}$, in which case

$$\alpha^2\|\lambda^A\|^2 = \frac{3}{\sqrt{2}}\int_\Sigma\Big(-t_{AA'}\big(\tilde{\mathcal{D}}_e\lambda^A\big)\big(\tilde{\mathcal{D}}^e\bar\lambda^{A'}\big) - \frac{1}{2}t^a\,{}^4G_{aBB'}\lambda^B\bar\lambda^{B'}\Big)\mathrm{d}\Sigma. \tag{4.4}$$

Remarkably enough, apart from the numerical coefficient $\frac{3}{\sqrt{2}}$ the right hand side is precisely the integral of the right hand side of (2.13), whose integral on an asymptotically flat Σ gave the appropriate component of the total energy-momentum of the localized matter+gravity systems. In fact, using the unitary spinor form $\mathcal{D}_{EF} := G_F{}^{E'}\mathcal{D}_{E'E} = \mathcal{D}_{(EF)}$ of the Sen derivative operator $\mathcal{D}_e$ the decomposition of the derivative $\mathcal{D}_e\lambda_A$ into its irreducible parts is

$$\begin{aligned}G_F{}^{E'}\mathcal{D}_{E'E}\lambda_A &= \mathcal{D}_{(EF}\lambda_{A)} + \frac{1}{3}\varepsilon_{EA}\mathcal{D}_{FB}\lambda^B + \frac{1}{3}\varepsilon_{FA}\mathcal{D}_{EB}\lambda^B = \\ &= \mathcal{D}_{(EF}\lambda_{A)} + \frac{1}{3}\varepsilon_{EA}G_F{}^{K'}\mathcal{D}_{K'K}\lambda^K + \frac{1}{3}\varepsilon_{FA}G_E{}^{K'}\mathcal{D}_{K'K}\lambda^K = \\ &= \mathcal{D}_{(EF}\lambda_{A)} + \frac{1}{3}G_F{}^{E'}\big(\varepsilon_{EA}\delta^{K'}_{E'} - G_E{}^{K'}G_{E'A}\big)\mathcal{D}_{K'K}\lambda^K = \\ &= \mathcal{D}_{(EF}\lambda_{A)} + \frac{2}{3}G_F{}^{E'}P^{KK'}_{EE'}\varepsilon_{KA}\mathcal{D}_{K'L}\lambda^L;\end{aligned} \tag{4.5}$$

and hence, taking into account that in the positive energy proofs $\mathcal{D}_{A'A}\lambda^a=0$,

$$-t_{AA'}\big(\tilde{\mathcal{D}}_e\lambda^A\big)\big(\tilde{\mathcal{D}}^e\bar\lambda^{A'}\big)=2t^{AA'}t^{BB'}t^{EE'}\big(\mathcal{D}_{(AB}\lambda_{E)}\big)\big(\mathcal{D}_{(A'B'}\bar\lambda_{E')}\big)= \qquad (4.6)$$
$$=-t_{AA'}\big(\mathcal{D}_e\lambda^A\big)\big(\mathcal{D}^e\bar\lambda^{A'}\big).$$

It might be interesting to note that $\mathcal{D}_{(AB}\lambda_{C)}$ is just the 3-surface twistor derivative of the spinor field [20, 21, 22]: $\mathcal{D}_{(AB}\lambda_{C)}=0$ is the purely spatial part in the complete irreducible 3+1 decomposition of the 1-valence spacetime twistor equation $\nabla_{A'(A}\lambda_{B)}=0$.

If we introduced the derivative $\tilde{\mathcal{D}}_e\lambda^A$ by the more general expression $\mathcal{D}_e\lambda^A+sP_e^{AA'}\mathcal{D}_{A'B}\lambda^B+FP_e^{AA'}t_{A'B}\lambda^B$ for some real constant s and complex function F, then in the integrand on the right hand side of (4.4) we would have the extra *negative definite* term $-\frac{3}{4}F\bar{F}t_{AA'}\lambda^A\bar\lambda^{A'}$. This term would *decrease* the right hand side of (4.4) (and the lower bound below), and hence its introduction does not seem to be useful.

Finally, by (4.4) we have the lower bound for the eigenvalues

$$\alpha^2\geq-\frac{3}{4}\inf\frac{\int_\Sigma t^a\,{}^4G_{ab}l^b\,\mathrm{d}\Sigma}{\int_\Sigma t_bl^b\,\mathrm{d}\Sigma}=\frac{3}{4}\kappa\inf\frac{\int_\Sigma t^aT_{ab}l^b\,\mathrm{d}\Sigma}{\int_\Sigma t_bl^b\,\mathrm{d}\Sigma}, \qquad (4.7)$$

where, as above, the infimum is taken on the set of the smooth, future pointing null vector fields l^a on Σ. The quotient of the integrals is some average on Σ of the flux of the energy current $T^a{}_bl^b$ of the matter fields seen by the null observer l^a. In the special case of the vanishing extrinsic curvature this bound is not less then Friedrich's sharp lower bound.

5 The limiting case

If the equality hols in (4.7), then by (4.4) and (4.6) the eigenspinor λ_A must also solve the 3-surface twistor equation $\mathcal{D}_{(AB}\lambda_{C)}=0$. Then the derivative of the spinor field λ_A can be expressed in terms of $\bar\mu_{A'}$ algebraically, and we can evaluate its integrability condition to obtain a condition on the geometry of the data set $(\Sigma,h_{ab},\chi_{ab})$. However, instead of the general analysis of this limiting case we show directly through an example that the lower bound (4.7) is sharp.

The example is the $t=$ const spacelike hypersurface in a $k=1$ Friedman–Robertson–Walker cosmological spacetime. Explicitly, the manifold Σ is homeomorphic to S^3, the intrinsic metric h_{ab} is the standard 3-sphere metric with scalar curvature $R=$ const, and the extrinsic curvature is $\chi_{ab}=\frac{1}{3}\chi h_{ab}$ with $\chi=$ const. For this data set $t^a\,{}^4G_{ab}P^b_c=0$ and $-t^at^b\,{}^4G_{ab}=\frac{1}{2}R+\frac{1}{3}\chi^2=$ const, and hence the lower bound (4.7) is $\frac{3}{8}R+\frac{1}{4}\chi^2$.

On the other hand, we know that this example with $\chi=0$ saturates the inequality of Friedrich, i.e. the smallest eigenvalue ${}_0\alpha^2_1$ of the (Riemannian) eigenvalue problem $2D^{AA'}D_{A'B}\lambda^B={}_0\alpha^2\lambda^A$ is just $\frac{3}{8}R$. We show that the corresponding eigenspinor is an eigenspinor of $2\mathcal{D}^{AA'}\mathcal{D}_{A'B}$ too, and the corresponding eigenvalue saturates (4.7). In fact, since $\chi=$ const, $2\mathcal{D}^{AA'}\mathcal{D}_{A'B}\lambda^B=$

$2D^{AA'}D_{A'B}\lambda^B + \frac{1}{4}\chi^2\lambda^A$ holds, and hence for the smallest eigenvalue α_1 of the Sen–Witten operator we obtain $\alpha_1^2 = {}_0\alpha_1^2 + \frac{1}{4}\chi^2$, just the lower bound coming from (4.7). The extrinsic curvature shifted both Friedrich's lower bound and the smallest Riemannian eigenvalue by the same positive term $\frac{1}{4}\chi^2$. It is easy to see that the 3-surface twistor operator also annihilates this eigenspinor: since it is annihilated by the Riemannian 3-surface twistor operator and $\mathcal{D}_{AB}\lambda_C = D_{AB}\lambda_C + \frac{1}{6\sqrt{2}}\chi(2\varepsilon_{BC}\lambda_A + \varepsilon_{AB}\lambda_C)$ holds, $\mathcal{D}_{(AB}\lambda_{C)} = 0$ follows.

This work was partially supported by the Hungarian Scientific Research Fund (OTKA) grant K67790.

References

[1] A. Lichnerowicz, Spineurs harmoniques, C. R. Acad. Sci. Paris A–B **257** 7–9 (1963)

[2] Th. Friedrich, Der erste Eigenwert des Dirac-operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung, Math. Nachr. **97** 117–146 (1980)

[3] O. Hijazi, A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors, Commun. Math. Phys. **104** 151–162 (1986)

[4] O. Hijazi, Lower bounds for the eigenvalues of the Dirac operator, J. Geom. Phys. **16** 27–38 (1995)

[5] Th. Friedrich, *Dirac Operators in Riemannian Geometry*, Graduate Studies in Mathematics, Vol 25, AMS Providence, Rhode Island 2000

[6] C. Bär, Lower eigenvalue estimates for Dirac operators, Math. Ann. **293** 39–46 (1992)

[7] Th. Friedrich, E. C. Kim, Some remarks on the Hijazi inequality and generalizations of the Killing equation for spinors, J. Geom. Phys. **37** 1–14 (2001)

[8] X. Zhang, Lower bounds for eigenvalues of hypersurface Dirac operators, Math. Res. Lett. **5** 199–210 (1998)

[9] R. Penrose, W. Rindler, *Spinors and Spacetime*, Vol 1, Cambridge University Press, Cambridge 1984

[10] A. Sen, On the existence of neutrino 'zero-modes' in vacuum spacetimes, J. Math. Phys. **22** 1781–1786 (1981)

[11] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, vol. 1, Interscience, New York, 1964

[12] L. B. Szabados, On the role of conformal three-geometries in the dynamics of general relativity, Class. Quantum Grav. **19** 2375–2391 (2002), gr-qc/0110106

[13] L. B. Szabados, Total angular momentum from Dirac eigenspinors, Class. Quantum Grav. (to appear), arXiv:0709.1072

[14] O. Reula, K. P. Tod, Positivity of the Bondi energy, J. Math. Phys. **25** 1004–1008 (1984)

[15] E. Witten, A new proof of the positive energy theorem, Commun. Math. Phys. **30** 381–402 (1981)

[16] J. M. Nester, A new gravitational energy expression and with a simple positivity proof, Phys. Lett. A, **83** 241–242 (1981)

[17] G. W. Gibbons, S. W. Hawking, G. T. Horowitz, M. J. Perry, Positive mass theorem for black holes, Commun. Math. Phys. **88** 295–308 (1983)

[18] O. Reula, Existence theorem for solutions of Witten's equation and nonnegativity of total mass, J. Math. Phys. **23** 810–814 (1982)

[19] J. Frauendiener, Triads and the Witten equation, Class. Quantum Gravity, **8** 1881–1187 (1991)

[20] K. P. Tod, Three-surface twistors and conformal embedding, Gen. Rel. Grav. **16** 435–443 (1984)

[21] R. Penrose, W. Rindler, *Spinors and Spacetime*, Vol 2, Cambridge University Press, Cambridge 1986

[22] L. B. Szabados, On certain global conformal invariant and 3-surface twistors of initial data sets, Class. Quantum Grav. **17** 793–811 (2000), gr-qc/9909052